Compendium of formulas regarding rectangular combinatorics

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1 Binomial coefficient (recalling classical properties)

Binomial relation:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Symmetry:

$$\binom{n}{k} = \binom{n}{n-k}$$

Pascal's relation:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Identities: **($f_n : n^{th}$-Fibonacci number)**

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n; \qquad \sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}; \qquad \sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0;$$

$$\sum_{i=k}^{n} \binom{n}{i}\binom{i}{k} = 2^{n-k}\binom{n}{k}; \qquad \sum_{i=0}^{n} i\binom{n}{i}^2 = \frac{n}{2}\binom{2n}{n}; \qquad \sum_{i=0}^{m} (-1)^i \binom{n}{i} = (-1)^m \binom{n-1}{m};$$

$$\sum_{i=0}^{n} \binom{i}{k} = \binom{n+1}{k+1}; \qquad \sum_{i=0}^{n} \binom{m+i}{i} = \binom{m+n+1}{n}; \qquad \sum_{i=0}^{n} \binom{n-i}{i} = f_{n+1};$$

$$\sum_{i=0}^{n} \binom{n}{i}\binom{m}{k-i} = \binom{n+m}{k}; \qquad \binom{n}{l}\binom{l}{k} = \binom{n}{k}\binom{n-k}{l-k}; \qquad \sum_{k=-n}^{n} (-1)^k \binom{2n}{k+n}^3 = \frac{(3n)!}{(n!)^3};$$

Multinomial coefficient: $\binom{n}{k_1\ k_2\ \dots\ k_m} = \frac{n!}{k_1!k_2!\dots k_m!}$, for $k_1 + k_2 + \dots + k_m = n$

Multinomial relation: $(x_1+x_2+\dots+x_m)^n = \sum_{k_1+k_2+\dots+k_m=n} \binom{n}{k_1\ k_2\ \dots\ k_m} x_1^{k_1} x_2^{k_2} \dots x_m^{k_m}$

2 q-analogs

A q-analog of $f_n \in \mathbb{N}$, is a polynomial $\bar{f}_n(q)$ in $\mathbb{N}[q]$ such that $\lim_{q\to 1} \bar{f}_n(q) = f_n$.

Classical q-analogs:

q-integer: $[n]_q := \frac{1-q^n}{1-q} = 1 + q + q^2 + \dots + q^{n-1};$

q-factorial: $[n]_q! := \prod_{i=1}^{n} [i]_q = [n]_q [n-1]_q ... [2]_q [1]_q;$

q-binomial coefficient: $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!} = \prod_{i=1}^{k} \frac{[n-i+1]_q}{[i]_q}$

q-binomial identities:

$$\prod_{i=0}^{n-1} 1 + xq^i = \sum_{k=0}^{n} q^{\binom{n}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k$$

$$\prod_{i=0}^{n-1} \frac{1}{1-xq^i} = \sum_{k=0}^{n} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q x^k$$

q-Pascal's relations

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^r \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$$

Identities: (We set $\chi(P)$ equal to 1 if P is true, and 0 otherwise.)

$$\sum_{i=0}^{n} (-1)^i \begin{bmatrix} n \\ k \end{bmatrix}_q = \chi(n \text{ even}) \prod_{i=1}^{k} 1 - q^{2i-1}; \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q;$$

Assuming $xy = qyx$, then $(x+y)^n = \sum_{i=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^i y^{n-i}$; $\sum_{i=0}^{n} q^{i(m-k+i)} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} m \\ k-i \end{bmatrix} = \begin{bmatrix} n+m \\ k \end{bmatrix}$;

As usual, we set:

$\mathrm{inv}(\sigma) := \#\{(i,j) \mid 1 \le i < j \le n \text{ and } \sigma(i) > \sigma(j)\}$; $\quad \mathrm{maj}(\sigma) := \sum_{\sigma(i)>\sigma(i+1)} i$;

$\mathrm{cycle}(\sigma) :=$ number of cycles in σ.

Some combinatorial formulas involving q-analogs:

$$\sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma \in S_n} q^{\mathrm{maj}(\sigma)} = [n]_q!; \qquad \sum_{\sigma \in S_n} q^{\mathrm{cycle}(\sigma)} = \prod_{i=0}^{n-1} (q+i);$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{m=0}^{n} a_m q^m; \qquad \text{where} \quad a_m = \sum_{\lambda \vdash m} \#\{k \in \mathbb{N} \mid \lambda_1 \le n-k \text{ and } \ell(\lambda) \le k\}$$

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3 Catalan numbers

Definition: $C_n = \frac{1}{n+1}\binom{2n}{n} = \prod_{i=2}^{n} \frac{n-i}{i} = \sum_{k=0}^{n-1} C_k C_{n-k-1} = \sum_{k=0}^{n-1} (-1)^k \binom{n-k}{k-1} C_k$ ($C_0 = C_1 = 1$)

Some combinatorial interpretations: (see Stanley's book for more)

$$\begin{aligned} C_n &= \#\{\text{triangulations of a } (n+2)\text{-gon}\} \\ &= \#\{w = w_1 w_2 ... w_n \in \{a,b\}^n \mid \forall i \; |w_1 w_2 ... w_i|_a \ge |w_1 w_2 ... w_i|_b\} \\ &= \#\{\text{full binary trees with } n+1 \text{ leaves}\} \\ &= \#\{\text{non-crossing partitions of an } n\text{-element set}\} \\ &= \#\{\text{Dyck paths on a } (n \times n)\text{-grid}\}(\text{see below}) \end{aligned}$$

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4 Lattice paths and Dyck paths

$$\begin{aligned}
\mathscr{L}_{m,n} &:= \{\gamma \subseteq \mathbb{N}^2 \mid \gamma \text{ goes from } (0,0) \text{ to } (m,n), \text{ by north or east steps}\} \\
\mathscr{C}_{m,n} &:= \{\gamma \in \mathscr{L}_{m,n} \mid \gamma \text{ stays weakly above the diagonal}\} \qquad ((m,n)\text{-Dyck paths}) \\
\mathscr{C}'_{m,n} &:= \{\gamma \in \mathscr{C}_{m,n} \mid \gamma \text{ stays strictly above the diagonal}\}
\end{aligned}$$

We set the notations $L_{m,n} := \#\mathscr{L}_{m,n}$, $C_{m,n} := \#\mathscr{C}_{m,n}$ and $C'_{m,n} := \#\mathscr{C}'_{m,n}$. See illustrations at the end.

Case m=n

$$C_{n,n} = C_n = \frac{1}{n+1}\binom{2n}{n}$$

Case gcd$(m,n) = 1$

$$C_{m,n} = \frac{1}{m+n}\binom{m+n}{n}$$

General case ($d := \mathbf{gcd}(m,n), a := m/d, b := n/d$)

$$C_{m,n} = \sum_{\mu \vdash d} \frac{1}{z_\mu} \prod_{k \in \mu} \frac{1}{a+b}\binom{ka+kb}{ka}.$$

For a partition $\mu = (\mu_1, \mu_2 ..., \mu_\ell) \vdash n$, with $d_i = \#\{k \mid \mu_k = i\}$, define $z_\mu := \prod_{i=1}^{n} i^{d_i} d_i!$.

Various equivalent descriptions of $\gamma \in \mathscr{C}_{m,n}$:

By steps:	$s(\gamma) = s_1 s_2 ... s_{n+m} \in \{N, E\}^*$	$N = (0,1)$, $E = (1,0)$, $s_i =$ the i^{th} step
By row lengths:	$a(\gamma) = (a_1, a_2, ..., a_n) \in \mathbb{N}^n$	$a_i =$ number of squares between the i^{th} N of γ and the diagonal
As a partition:	$\lambda(\gamma) = (\lambda_1, \lambda_2, ..., \lambda_n) \vdash \lvert\lambda\rvert$	$\lambda_i =$ number of E before the $(n-i+1)^{th}$ N
As a composition:	$\rho(\gamma) = (r_1, r_2, ..., r_k) \models n$	$r_i =$ length of the i^{th} block of N

Definition of various statistics: (i.e. functions $\mathscr{C}_{m,n} \longrightarrow \mathbb{N}$**)**

$$\begin{aligned}
\mathbf{inv}(\gamma) &:= \#\{(i,j) \mid 1 \le i < j \le n+m,\ s_i = E \text{ and } s_j = N\}; \\
\mathbf{maj}(\gamma) &:= \sum_{s_i = E\ ,\ s_{i+1} = N} i; \\
\mathbf{coinv}(\gamma) &:= \#\{(i,j) \mid 1 \le i < j \le n+m,\ s_i = N \text{ and } s_j = E\}; \\
\mathbf{area}(\gamma) &:= \sum_{i=1}^{n} a_i; \\
\mathbf{dinv}(\gamma) &:= \#\{(i,j) \mid 1 \le i < j \le n \text{ and } a_i = a_j\} + \#\{(i,j) \mid 1 \le i < j \le n \text{ and } a_i = a_j + 1\}
\end{aligned}$$

Formulas:

$$\sum_{\gamma \in \mathscr{L}_{m,n}} q^{\mathrm{inv}(\gamma)} = \sum_{\gamma \in \mathscr{L}_{m,n}} q^{\mathrm{coinv}(\gamma)} = \sum_{\gamma \in \mathscr{L}_{m,n}} q^{\mathrm{maj}(\gamma)} = \begin{bmatrix} m+n \\ n \end{bmatrix}_q$$

$$\sum_{\gamma \in \mathscr{C}_{m,n}} q^{\mathrm{area}(\gamma)} = \sum_{\gamma \in \mathscr{C}_{m,n}} q^{\mathrm{dinv}(\gamma)}$$

Denote by $C^{(j)}_{m,n}$ the cardinality of the set $\mathscr{C}^{(j)}_{m,n} := \{\gamma \in \mathscr{C}_{m,n} \mid\ s_i = N \text{ and } s_{i+1} = E \text{ occurs exactly } j \text{ times}\}$.
Given any symmetric function $f = \sum_{\lambda \vdash n} a_\lambda \frac{p_\lambda}{z_\lambda}$, we have

$$\mathscr{N}_{m,n}(z) = \sum_{j=1}^{\min(m,n)} \frac{1}{m}\binom{m}{j}\binom{n-1}{j-1} z^j \qquad \mathscr{N}_{(m,n),f}(z) = \sum_{\mu \vdash d} \frac{a_\mu}{z_\mu} \prod_{k \in \mu} k \mathscr{N}_{ka,kb}(z)$$

$$\mathscr{N}_{(m,n),h_d}(z) = \sum_{j=1}^{\min(m,n)} C^{(j)}_{m,n}\, z^j \qquad \mathscr{N}_{(m,n),(-1)^{d-1}e_d}(z) = \sum_{j=1}^{\min(m,n)} {C'}^{(j)}_{m,n}\, z^j$$

The coefficients of $\mathscr{N}_{m,n}(z)$ are called the Narayana numbers.

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5 (q,t)-Catalan numbers and Dyck paths

q,t-Catalan numbers:

$$\begin{aligned} C_n(q,t) &:= \sum_{\gamma \in \mathscr{C}_{n,n}} q^{\mathrm{dinv}(\gamma)} t^{\mathrm{area}(\gamma)} \\ C_n(q,t) &= C_n(t,q) \quad \text{(no direct bijection known)} \end{aligned}$$

q-Catalan numbers:

$$\begin{aligned} C_n(q) &:= q^{\binom{n}{2}} C_n(q, \tfrac{1}{q}) = \sum_{\gamma \in \mathscr{C}_{m,n}} q^{\mathrm{maj}(\gamma)} = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q \\ \widetilde{C}_n(q) &:= C_n(q,1) = \sum_{\gamma \in \mathscr{C}_{m,n}} q^{\mathrm{area}(\gamma)} \end{aligned}$$

Identities:

$$\begin{aligned} C_n(q) &= \sum_{k=1}^{n} (-1)^{k-1} q^{r^2-2} \left(\prod_{i=1}^{k} \frac{1+q^{n-k+1+i}}{1+q^i} \right) \begin{bmatrix} n-k+1 \\ k \end{bmatrix}_q C_k(q) \\ \widetilde{C}_n(q) &= \sum_{k=0}^{n-1} q^k \widetilde{C}_k(q) \widetilde{C}_{n-k-1}(q) \end{aligned}$$

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6 Parking functions

We respectively denote by $\mathscr{P}^{(\gamma)}_{m,n}$, $\mathscr{P}_{m,n}$, and $\mathscr{P'}_{m,n}$ the sets of "parking functions of shape γ", "(m,n)-parking functions", and "diagonal avoiding (m,n)-parking functions". These are defined as

$$\begin{aligned} \mathscr{P}^{(\gamma)}_{m,n} &= \{\lambda = (\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, ..., \lambda_{\sigma(n)}) \in \mathbb{N}^n \mid \lambda(\gamma) = (\lambda_1, \lambda_2, ..., \lambda_n), \sigma \in S_n\} \\ \mathscr{P}_{m,n} &= \bigcup_{\gamma \in \mathscr{C}_{m,n}} \mathscr{P}^{(\gamma)}_{m,n} \\ \mathscr{P'}_{m,n} &= \bigcup_{\gamma \in \mathscr{C'}_{m,n}} \mathscr{P}^{(\gamma)}_{m,n} \end{aligned}$$

Their respective cardinalities are denoted: $P^{(\gamma)}_{m,n} := \#\mathscr{P}^{(\gamma)}_{m,n}$, $P_{m,n} := \#\mathscr{P}_{m,n}$, and $\#P'_{m,n} = \mathscr{P'}_{m,n}$. We have

$$P^{(\gamma)}_{m,n} = \binom{n}{\rho(\gamma)}; \qquad P_{m,n} = \sum_{\gamma\in\mathscr{C}_{m,n}} P^{(\gamma)}_{m,n} = \sum_{\gamma\in\mathscr{C}_{m,n}} \binom{n}{\rho(\gamma)}$$

Action of $\mathbb{S}_n$ on $\mathscr{P}^{(\gamma)}_{m,n}$:	$\sigma\cdot\pi = (\pi_{\sigma^{-1}(1)}, \pi_{\sigma^{-1}(2)}, ..., \pi_{\sigma^{-1}(n)})$
Frobenius characterisitic of $\mathscr{P}^{(\gamma)}_{m,n}$:	$\mathscr{P}^{(\gamma)}_{m,n}(\boldsymbol{x}) = h_{\rho(\gamma)}(\boldsymbol{x})$
Frobenius characterisitic of $\mathscr{P}_{m,n}$:	$\mathscr{P}_{m,n}(\boldsymbol{x}) = \sum_{\gamma\in\mathscr{C}_{m,n}} h_{\rho(\gamma)}(\boldsymbol{x})$
Frobenius characterisitic of $\mathscr{P}'_{m,n}$:	$\mathscr{P}'_{m,n}(\boldsymbol{x}) = \sum_{\gamma\in\mathscr{C}'_{m,n}} h_{\rho(\gamma)}(\boldsymbol{x})$

As before, we set $d := \gcd(m,n)$, $a = m/d$, $b := n/d$, and consider the linear and multiplicative operator $\Theta_{a,b}$ on symmetric functions, such that

$$\begin{aligned}\Theta_{a,b}(p_k) &:= \frac{1}{a} h_{bk}[ak\,\boldsymbol{x}] \\ &= \frac{1}{a}\sum_{\lambda\vdash bk} \frac{(ak)^{\ell(\lambda)}}{z_\lambda}\, p_\lambda(\boldsymbol{x})\end{aligned}$$

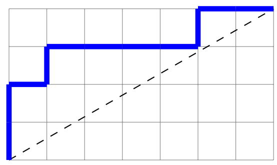
where we use plethystic notation assuming that a and b behave as constants.

Case $m=n$	**Case** $\gcd(m,n)=1$	**General case**
$\mathscr{P}_{n,n}(\boldsymbol{x}) = \sum_{\lambda\vdash n}(n+1)^{\ell(\lambda)-1}\frac{p_\lambda(\boldsymbol{x})}{z_\lambda}$	$\mathscr{P}_{m,n}(\boldsymbol{x}) = \frac{1}{m}\sum_{\lambda\vdash n} m^{\ell(\lambda)}\frac{p_\lambda(\boldsymbol{x})}{z_\lambda}$	$\mathscr{P}_{m,n}(\boldsymbol{x}) = \Theta_{a,b}(h_d(\boldsymbol{x}))$
$\mathscr{P}'_{n,n}(\boldsymbol{x}) = \mathscr{P}_{n,n-1}(\boldsymbol{x})$	$\mathscr{P}'_{m,n}(\boldsymbol{x}) = \mathscr{P}_{m,n}(\boldsymbol{x})$	$\mathscr{P}'_{m,n}(\boldsymbol{x}) = \Theta_{a,b}((-1)^{d-1}e_d(\boldsymbol{x}))$
$P_{n,n} = (n+1)^{n-1}$	$P_{m,n} = m^{n-1}$	$P_{m,n} =< \mathscr{P}_{m,n}(\boldsymbol{x}), h_1^n >$

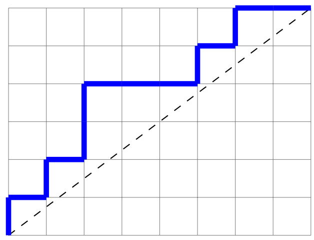
where $< \cdot, \cdot >$ is the usual scalar product on symmetric functions. We also have

$$\sum_{\gamma\in\mathscr{C}_{m,n}} e_{\rho(\gamma)}(\boldsymbol{x}) = \sum_{\mu\vdash d}\frac{1}{z_\mu}\prod_{k\in\mu}\frac{1}{a}\, e_{kb}[ak\,\boldsymbol{x}]$$

Illustrations involving Dyck paths



γ_1, a $(7,4)$-Dyck path



γ_2, a $(8,6)$-Dyck path

$$\begin{aligned} s(\gamma_1) &= NNENEEEEENEE \\ a(\gamma_1) &= (0,1,2,0) \\ \lambda(\gamma_1) &= (5,1,0,0) \\ \rho(\gamma_1) &= (1,1,2) \\ \mathrm{inv}(\gamma_1) &= 2+1+1+1+1 \\ &= 6 \\ \mathrm{maj}(\gamma_1) &= 3+8 \\ &= 11 \\ \mathrm{coinv}(\gamma_1) &= 7+7+6+2 \\ &= 22 \\ \mathrm{area}(\gamma_1) &= 0+1+2+0 \\ &= 3 \\ \mathrm{dinv}(\gamma_1) &= 1+1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} s(\gamma_2) &= NENENNEEENENEE \\ a(\gamma_2) &= (0,0,0,2,0,0) \\ \lambda(\gamma_2) &= (6,5,2,2,1,0) \\ \rho(\gamma_2) &= (1,1,2,1,1) \\ \mathrm{inv}(\gamma_2) &= 5+4+2+2+2+1 \\ &= 16 \\ \mathrm{maj}(\gamma_2) &= 2+4+9+11 \\ &= 26 \\ \mathrm{coinv}(\gamma_2) &= 8+7+6+6+3+2 \\ &= 32 \\ \mathrm{area}(\gamma_2) &= 0+0+0+2+0+0 \\ &= 2 \\ \mathrm{dinv}(\gamma_2) &= 4+3+2+1 \\ &= 10 \end{aligned}$$

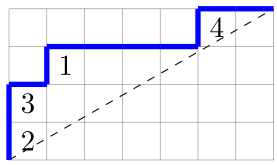
$$\gcd(7,4) = 1$$

$$\begin{aligned} C_{7,4} &= \frac{1}{7+4}\binom{7+4}{7} \\ &= \frac{(10)(9)(8)}{4} \\ &= 180 \end{aligned}$$

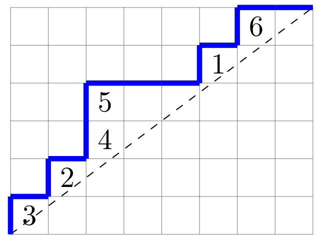
$$\gcd(8,6) = 2$$

$$\begin{aligned} C_{8,6} &= \frac{1}{z_{(2)}}\left(\frac{1}{4+3}\binom{8+6}{8}\right) \\ &\quad + \frac{1}{z_{(1,1)}}\left(\frac{1}{4+3}\binom{4+3}{4}\right)^2 \\ &= \frac{429}{2} + \frac{25}{2} \\ &= 227 \end{aligned}$$

Illustrations involving parking functions



π_1, a $(7,4)$-parking function of shape γ_1



π_2, a $(8,6)$-parking function of shape γ_2

$$\begin{aligned}\lambda(\gamma_1) &= (5,1,0,0)\\ \pi_1 &= (0,5,1,0)\\ &= (3124)\cdot\lambda(\gamma_1)\\ &= (4123)\cdot\lambda(\gamma_1)\\ &= \{\{4\},\{1\},\{2,3\}\}\end{aligned}$$

$$\begin{aligned}\lambda(\gamma_2) &= (6,5,2,2,1,0)\\ \pi_2 &= (0,6,2,1,5,2)\\ &= (613524)\cdot\lambda(\gamma_2)\\ &= (614523)\cdot\lambda(\gamma_2)\\ &= \{\{6\},\{1\},\{4,5\},\{2\},\{3\}\}\end{aligned}$$

$$\begin{aligned}\mathscr{P}_{7,4} &= 7^{4-1}\\ &= 343\end{aligned}$$

$$\begin{aligned}\mathscr{P}_{8,6} &=< \mathscr{P}_{8,6}(\mathbf{x}), h_1^6(\mathbf{x}) >\\ &= 35328\end{aligned}$$

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