

# MATHEMAGICAL FORMULAS FOR SYMMETRIC FUNCTIONS

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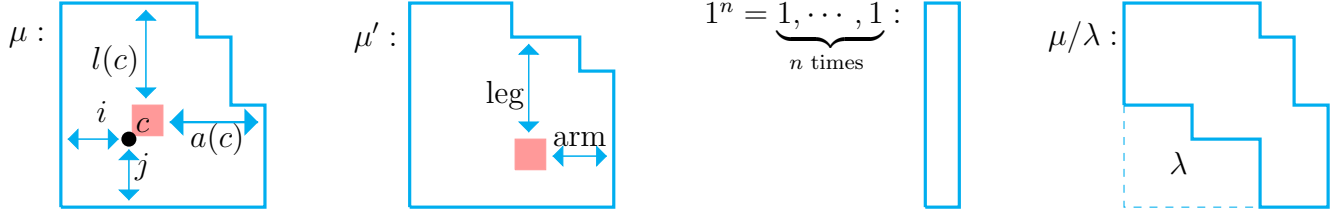
## THE BIBLIOGRAPHY

### SEE ALSO

- [a] [https://en.wikipedia.org/wiki/Partition\\_\(number\\_theory\)](https://en.wikipedia.org/wiki/Partition_(number_theory))
- [b1] [https://en.wikipedia.org/wiki/Symmetric\\_polynomial](https://en.wikipedia.org/wiki/Symmetric_polynomial)
- [b2] [https://en.wikipedia.org/wiki/Ring\\_of\\_symmetric\\_functions](https://en.wikipedia.org/wiki/Ring_of_symmetric_functions)
- [c] [https://en.wikipedia.org/wiki/Young\\_tableau](https://en.wikipedia.org/wiki/Young_tableau)

## Partitions

$\mu \vdash n$  iff  $\mu = \mu_1, \dots, \mu_k$ ;  $\mu_1 \geq \dots \geq \mu_k > 0$ ; and  $n = \sum \mu_j := |\mu|$ .  $\ell(\mu) = k$ .



$$l_\mu(c) = l(c) := \mu'_{i+1} - (j + 1); \quad a_\mu(c) = a(c) := \mu_{j+1} - (i + 1); \quad h_\mu(c) = h(c) := a(c) + l(c) + 1$$

$$(1) \quad z_\mu := \prod_{i=1}^n i^{d_i} d_i! \quad \text{for } \mu = 1^{d_1} \dots n^{d_n} \quad (2) \quad n(\mu) := \sum_{c \in \mu} l(c) = \sum_{(i,j) \in \mu} j \quad \text{and} \quad n(\mu') := \sum_{c \in \mu} a(c) = \sum_{(i,j) \in \mu} i$$

[Example 1 clic here](#) [Wikipedia page on this](#)

## Tableaux

$\mu = \mu_1, \dots, \mu_k \vdash n$  iff  $\mu \subset \mathbb{N} \times \mathbb{N}$ ;  $\mu = \{c \mid c = (i, j), 0 \leq j \leq \ell(\mu) - 1; 0 \leq i \leq \mu_{j+1} - 1\}$ ;  $\sum \mu_j = n$ .

## Tableau

$$\tau : \mu \rightarrow \{1, 2, \dots, n\}$$

## Semi-Standard Tableau

$$\tau(a, j) < \tau(b, j) \Rightarrow a \leq b$$

$$\text{and } \tau(i, c) < \tau(i, d) \Rightarrow c < d$$

Standard Tableau  $f^\mu$ 

$$\tau \text{ bijection, } \tau(a, j) < \tau(b, j) \Rightarrow a < b$$

$$\text{and } \tau(i, c) < \tau(i, d) \Rightarrow c < d$$

## Hook lenght formula :

$$(3) \quad f^\mu = \frac{n!}{\prod_{c \in \mu} h(c)}$$

$$(4) \quad \sum_{\mu \vdash n} (f^\mu)^2 = n!$$

## Determinantal formula :

$$(5) \quad f^\mu = n! \det \left( \left( \frac{1}{(\mu_i - i + j)!} \right)_{i,j} \right)$$

[Example 2 clic here](#) [Wikipedia page on this](#)

2. CLASSICAL BASIS OF  $\Lambda$ 

$\Lambda = \Lambda_{\mathbb{Q}}$  : ring of symmetric functions;  $\mathbf{x} := \{x_1, x_2, x_3, \dots\}$ . Basis are indexed by partitions,  $g = g(\mathbf{x})$ .

## Monomial symmetric functions

$$(6) \quad m_\mu := \sum_{\substack{i_1, \dots, i_k \in \mathbb{N}^* \\ \text{distinct}}} x_{i_1}^{\mu_1} x_{i_2}^{\mu_2} \dots x_{i_k}^{\mu_k}$$

## Power sum symmetric functions :

$$(7) \quad p_n := \sum_{i \in \mathbb{N}} x_i^n = m_{(n)} \quad \text{and} \quad p_\mu := p_{\mu_1} \dots p_{\mu_k}$$

## Complete homogeneous symmetric functions Elementary symmetric functions

$$(8) \quad h_n := \sum_{\lambda \vdash n} m_\lambda \quad \text{and} \quad h_\mu := h_{\mu_1} \dots h_{\mu_k} \quad (9) \quad e_n := \sum_{i_1 < \dots < i_n} x_{i_1} \dots x_{i_n} = m_{1^n} \quad \text{and} \quad e_\mu := e_{\mu_1} \dots e_{\mu_k}$$

where  $h_0 = e_0 = 1$  and  $e_k = h_k = 0$  for all  $k < 0$ .

## Schur symmetric functions (Jacobi-Trudi determinant formulas)

$$(10) \quad s_\mu = \det((h_{\mu_i - i + j})_{i,j}); \quad s_{(n)} = h_n \quad (11) \quad s_{\mu'} = \det((e_{\mu_i - i + j})_{i,j}); \quad s_{1^n} = m_{1^n} = e_n$$

[Example 3 clic here](#) [Wikipedia page on this](#)

## 3. GENERATING FUNCTIONS AND IDENTITIES

**Generating functions**

$$(12) \quad E(t) := \sum_{r \geq 0} e_r t^r = \prod_{i \geq 1} (1 + x_i t) \quad \left. \vphantom{E(t)} \right\} \Rightarrow (14) \quad \sum_{r=0}^n (-1)^r e_r h_{n-r} = 0$$

$$(13) \quad H(t) := \sum_{r \geq 0} h_r t^r = \prod_{i \geq 1} \frac{1}{(1 - x_i t)} = \frac{1}{E(-t)}$$

$$(15) \quad P(t) := \sum_{r \geq 0} p_r t^r = \frac{d}{dt} \log H(t) = \frac{H'(t)}{H(t)} \Leftrightarrow H(t) = e^{P(t)} \quad \Rightarrow (16) \quad n h_n = \sum_{r=1}^n p_r h_{n-r}$$

$$(17) \quad P(-t) := \sum_{r \geq 0} p_r t^r = \frac{d}{dt} \log E(t) = \frac{E'(t)}{E(t)} \Leftrightarrow E(t) = e^{-P(-t)} \quad \Rightarrow (18) \quad n e_n = \sum_{r=1}^n (-1)^{r-1} p_r e_{n-r}$$

[Wikipedia page on this](#)

**Changing basis**

$$(15) \Rightarrow (19) \quad h_n(\mathbf{x}) = \sum_{\mu \vdash n} \frac{p_\mu(\mathbf{x})}{z_\mu} \quad (17) \Rightarrow (20) \quad e_n(\mathbf{x}) = \sum_{\mu \vdash n} \frac{(-1)^{n-\ell(\mu)} p_\mu(\mathbf{x})}{z_\mu}, \quad \text{see also 29 and 30}$$

**The  $\omega$  linear map**

$$\omega : \Lambda \rightarrow \Lambda, \quad \left. \vphantom{\omega} \right\} \Rightarrow \left\{ \begin{array}{l} \omega^2(g(\mathbf{x})) = g(\mathbf{x}), \quad \forall g \in \Lambda; \quad (21) \\ \omega(p_n) \mapsto (-1)^{n-1} p_n \end{array} \right.$$

**Changing basis**

$$\omega(h_n) = e_n; \quad (22) \quad \omega(m_\mu) = f_\mu, \{f_\mu\} \text{ is the forgotten base}$$

**Scalar product**

$$(23) \quad \langle p_\mu, p_\lambda \rangle := z_\mu \delta_{\mu, \lambda} \\ \langle g, d \rangle = \langle \omega(g), \omega(d) \rangle \quad \forall d, g \in \Lambda.$$

**Cauchy Kernel**

$$(24) \quad \Omega(\mathbf{x}\mathbf{y}) := \prod_{i \geq 1} \frac{1}{(1 - x_i y_i)}$$

 **$\{f_\mu\}$  and  $\{g_\mu\}$  dual basis iff**

$$\langle d_\mu, g_\lambda \rangle = \delta_{\mu, \lambda} \quad \text{or} \\ \Omega(\mathbf{x}\mathbf{y}) = \sum_{\mu} d_\mu(\mathbf{x}) g_\mu(\mathbf{y})$$

$$(25) \quad \Omega(\mathbf{x}\mathbf{y}) = \sum_n h_n(\mathbf{x}\mathbf{y}) = \sum_{\mu} s_{\mu}(\mathbf{x}) s_{\mu}(\mathbf{y}) = \sum_{\mu} h_{\mu}(\mathbf{x}) m_{\mu}(\mathbf{y}) = \sum_{\mu} e_{\mu}(\mathbf{x}) f_{\mu}(\mathbf{y}) = \sum_{\mu} p_{\mu}(\mathbf{x}) \frac{p_{\mu}(\mathbf{y})}{z_{\mu}}$$

## 4. FROBENIUS TRANSFORM AND HILBERT SERIES

**Cyclic structure**

$$\lambda(\sigma) = \lambda_1, \dots, \lambda_k \text{ iff } \sigma = (\sigma_1, \dots, \sigma_{\lambda_1}) \cdots (\sigma_{\lambda_1 + \dots + \lambda_{k-1} + 1}, \dots, \sigma_{|\lambda|})$$

**Cyclic type**

$$\sigma = \sigma_{\mu} \text{ iff } \lambda(\sigma) = \mu$$

**Class functions**

$$(26) \quad R(\mathbb{S}_n) := \{\chi : \mathbb{S}_n \rightarrow \mathbb{C} \mid \chi(\sigma) = \chi(\tau\sigma\tau^{-1}), \forall \tau \in \mathbb{S}_n\}; \quad \chi_V + \chi_W = \chi_{V \oplus W} \quad \text{and} \quad \chi_V \chi_W = \chi_{V \otimes W}$$

**Characters****Frobenius transform  $\mathcal{F}$  (of an  $\mathbb{S}_n$ -module  $V$ )**

$$(27) \quad \mathcal{F}(V) = \mathcal{F}(\chi_V) := \frac{1}{n!} \sum_{\sigma \in \mathbb{S}_n} \frac{1}{n!} \chi_V(\sigma) p_{\lambda(\sigma)} = \sum_{\lambda \vdash n} \frac{1}{z_{\lambda}} \chi_V(\sigma_{\lambda}) p_{\lambda} \quad \Rightarrow \quad \mathcal{F}(V \oplus W) = \mathcal{F}(V) + \mathcal{F}(W)$$

**Changing basis**

$$(28) \quad \mathcal{F}(\chi^\mu) = \sum_{\lambda \vdash n} \frac{1}{z_\mu} \chi^\mu(\sigma_\lambda) p_\lambda = s_\mu,$$

where  $\chi^\mu$  is an irreducible character.

$$(29) \quad \mathcal{F}(\chi_{1_{\mathbb{S}_n}}) = \sum_{\lambda \vdash n} \frac{1}{z_\mu} p_\lambda = h_n,$$

where  $1_{\mathbb{S}_n}$  is the trivial representation.

$$(30) \quad \mathcal{F}(\chi_{\text{Sign}_{\mathbb{S}_n}}) = \sum_{\lambda \vdash n} \frac{1}{z_\mu} \chi_{\text{Sign}_{\mathbb{S}_n}}(\sigma_\lambda) p_\lambda = \sum_{\mu \vdash n} \frac{1}{z_\mu} (-1)^{n-\ell(\mu)} p_\mu = e_n,$$

Note 29 is equivalent to 19 and 30 is equivalent to 20.

where  $\text{Sign}_{\mathbb{S}_n}$  is the sign representation.

**Graded Frobenius characteristic**

$$(31) \quad \text{Frob}_q(V) := \sum_{n \geq 1} \mathcal{F}(V_n) q^n,$$

where  $V = \bigoplus_{n \geq 1} V_n$  is a graded  $\mathbb{S}_n$ -module.

**Bigraded Frobenius characteristic**

$$(32) \quad \text{Frob}_{q,t}(V) := \sum_{n \geq 1} \mathcal{F}(V_{n,k}) q^n t^k,$$

where,  $V = \bigoplus_{n,k > 1} V_{n,k}$  is a bigraded  $\mathbb{S}_n$ -module.

**Hilbert series (poincaré series)**

$$(33) \quad \text{Hilb}_q(V) := \sum_{n \geq 1} \dim(V_n) q^n,$$

where  $V = \bigoplus_{n \geq 1} V_n$  is a graded space.

**Bigraded Hilbert series**

$$(34) \quad \text{Hilb}_{q,t}(V) := \sum_{n \geq 1} \dim(V_{n,k}) q^n t^k,$$

where,  $V = \bigoplus_{n,k \geq 1} V_{n,k}$  is a bigraded space.

5. PLETHYSM ( $\lambda$ -RINGS)

**plethysm** is defined by :

$$(35) \quad p_n[\mathbf{x} + Y] = p_n[\mathbf{x}] + p_n[Y], \quad (37) \quad p_n[x] = x^n \text{ therefor } p_n[p_k(\mathbf{x})] = p_{nk}(\mathbf{x}), \quad p_n[q\mathbf{x}] = q^n p_n(\mathbf{x})$$

$$(36) \quad p_n[\mathbf{x}Y] = p_n[\mathbf{x}]p_n[Y] \quad (38) \quad p_n[c] = c, \text{ if } c \text{ is a constant,} \quad p_n[t\mathbf{x}] = t^n p_n(\mathbf{x})$$

[Example 4 clic here](#)

6. MACDONALD SYMMETRIC FUNCTIONS

**More scalar product**

... for original Macdonald polynomials

...for combinatorial Macdonal polynomials

$$(39) \quad \langle p_\mu, p_\lambda \rangle_{q,t} = z_\mu \delta_{\lambda,\mu} \prod_{i=1}^{\ell(\mu)} \frac{1 - q^{\mu_i}}{1 - t^{\mu_i}}$$

$$(40) \quad \langle p_\mu, p_\lambda \rangle_* = (-1)^{|\mu| - \ell(\mu)} z_\mu \delta_{\lambda,\mu} \prod_{i=1}^{\ell(\mu)} (1 - q^{\mu_i})(1 - t^{\mu_i})$$

$$(41) \quad \langle H_\mu, H_\lambda \rangle_* = \mathcal{E}_\mu(q, t) \mathcal{E}'_\mu(q, t) \delta_{\lambda,\mu}, \text{ where } \mathcal{E}_\mu(q, t) = \prod_{c \in \mu} (q^{a(c)} - t^{\ell(c)+1}) \text{ and } \mathcal{E}'_\mu(q, t) = \prod_{c \in \mu} (t^{\ell(c)} - q^{a(c)+1})$$

**Cauchy formula for the  $H_\mu$**

$$(42) \quad e_n \left[ \frac{\mathbf{x}\mathbf{y}}{(1-q)(1-t)} \right] = \sum_{\mu \vdash n} \frac{H_\mu(\mathbf{x}; q, t) H_\mu(\mathbf{y}; q, t)}{\mathcal{E}_\mu(q, t) \mathcal{E}'_\mu(q, t)}$$

**Original Macdonald polynomials**

(Gram-Schmidt of the monomial basis  
in respect to  $\langle \cdot, \cdot \rangle_{q,t}$ )

$$(43) \quad P_\mu(\mathbf{x}; q, t) = m_\mu + \sum_{\gamma \prec \mu} u_\gamma(q, t) m_\gamma$$

$$(45) \quad H_\mu(\mathbf{x}; q, 1) = \prod_i H_{\mu_i}(\mathbf{x}; q, 1)$$

$$(47) \quad H_n(\mathbf{x}; q, 1) = h_n \left[ \frac{\mathbf{x}}{1-q} \right] \prod_{i=1}^n (1 - q^i)$$

$$(49) \quad H_\mu(\mathbf{x}; q, t) = \text{Frob}_{q,t}(\mathcal{M}_\mu), \text{ where } \mathcal{M}_\mu = \mathbb{C}\{\delta \mathbf{x}^\alpha \delta \mathbf{y}^\beta \Delta_\mu(\mathbf{x}, \mathbf{y}) \mid \alpha, \beta \in \mathbb{N}^n\} \text{ is a Garcia-Haiman module and } \Delta_\mu = \det(x_k^i y_k^j)_{\substack{1 \leq k \leq n \\ (i,j) \in \mu}}.$$

**Specialisation**

$$(50) \quad H_u(\mathbf{x}; 0, 0) = s_n \quad (51) \quad H_u(\mathbf{x}; 0, 1) = h_u \quad (52) \quad H_u(\mathbf{x}; 1, 1) = s_1^n$$

 **$(q, t)$ -Kostka polynomials  $K_{\lambda, \mu}(q, t)$** 

$$(53) \quad H_\mu(\mathbf{x}; q, t) = \sum_{\lambda \vdash |\mu|} K_{\lambda, \mu}(q, t) s_\lambda(\mathbf{x}), \text{ where } K_{\lambda, \mu}(q, t) \in \mathbb{N}[q, t]. \quad (54) \quad K_{\lambda, \mu}(q, t) = K_{\lambda, \mu'}(t, q)$$

$$(55) \quad K_{\lambda, \mu}(q, t) = q^{n(\mu')} t^{n(\mu)} K_{\lambda', \mu}(q^{-1}, t^{-1}) \quad (56) \quad K_{\lambda, \mu}^{-1}(t, q) = K_{\lambda', \mu}^{-1}(q, t)$$

## 7. MACDONALD OPERATORS

**The  $\nabla$  operator**

$$(57) \quad \nabla(H_\mu) := q^{n(\mu')} t^{n(\mu)} H_\mu \quad (58) \quad \nabla(\Lambda_{\mathbb{Z}[q,t]}) \subseteq \Lambda_{\mathbb{Z}[q,t]} \text{ and } \nabla^{-1}(\Lambda_{\mathbb{Z}[q,t]}) \subseteq \Lambda_{\mathbb{Z}[q,t,1/q,1/t]}$$

$$(59) \quad \nabla(e_n) = \text{Frob}_{q,t}(\mathcal{DH}_n), \text{ where } \mathcal{DH} = \{f \in \mathbb{C}[\mathbf{x}, \mathbf{y}] \mid p_{h,k}(\delta \mathbf{x}, \delta \mathbf{y}) f(\mathbf{x}, \mathbf{y}) = 0, \forall h, k \text{ s.t. } h + k > 0\}$$

is the diagonal harmonic space.

$$(60) \quad \nabla(e_n)|_{t=1} = \sum_{\gamma \in \mathcal{D}_{n,n}} q^{\text{area}(\gamma)} e_{\rho(\gamma)}, \text{ see figure 2.} \quad (61) \quad \langle \nabla(e_n), e_n \rangle = C_n(q, t)$$

[Example 6 clic here](#)

**The  $\Delta_F$  operators**

$$(62) \quad B_\mu := \sum_{(i,j) \in \mu} q^i t^j \quad (63) \quad \Delta_F H_\mu(\mathbf{x}; q, t) := F[B_\mu] H_\mu(\mathbf{x}; q, t) \quad (64) \quad \Delta_F(\Lambda_{\mathbb{Z}[q,t]}) \subseteq \Lambda_{\mathbb{Z}[q,t]}$$

$$(65) \quad \Delta_{FG} = \Delta_F \circ \Delta_G \quad (66) \quad \Delta_{F+G} = \Delta_F + \Delta_G \quad (67) \quad \Delta_{cG} = c \Delta_G, \text{ for } c \in \mathbb{Q}$$

 **$\omega^*$  and  $\omega$** 

$$(68) \quad \omega^*(F(\mathbf{x}; q, t)) := \omega(F(\mathbf{x}; q^{-1}, t^{-1})) \quad (69) \quad \omega^*(H_\mu(\mathbf{x}; q, t)) = q^{-n(\mu')} t^{-n(\mu)} H_\mu(\mathbf{x}; q, t)$$

$$(70) \quad \omega^* \nabla \omega^*(H_\mu(\mathbf{x}; q, t)) = \nabla^{-1}(H_\mu(\mathbf{x}; q, t)) \quad (71) \quad \omega(H_\mu(\mathbf{x}; q, t)) = q^{n(\mu')} t^{n(\mu)} H_\mu(\mathbf{x}; q^{-1}, t^{-1})$$

$$(72) \quad \langle \nabla_{e_{d-1}}(e_n), F \rangle = \langle \nabla_{\omega F}(e_d), s_d \rangle, \forall F \in \Lambda^n \Rightarrow \quad \forall \mu \vdash n, \langle \nabla_{e_{d-1}}(e_n), s_\mu \rangle = \langle \nabla_{s_{\mu'}}(e_d), s_d \rangle,$$

The operator that multiplies by  $e_1$

The operator that differentiates by  $e_1$

$$(73) \quad \underline{e}_1 : \Lambda_{\mathbb{Q}(q,t)}^d \rightarrow \Lambda_{\mathbb{Q}(q,t)}^{d+1} \\ H_\mu \mapsto \sum_{\mu < \lambda} d_{\lambda,\mu}(q, t) H_\lambda$$

$$(74) \quad \delta_{e_1} : \Lambda_{\mathbb{Q}(q,t)}^d \rightarrow \Lambda_{\mathbb{Q}(q,t)}^{d-1} \\ H_\mu \mapsto \sum_{\lambda < \mu} c_{\lambda,\mu}(q, t) H_\lambda$$

$$d_{\lambda,\mu}(q, t) = \prod_{c \in \mathcal{R}_{\lambda,\mu}} \frac{q^{a_\mu(c)} - t^{l_\mu(c)+1}}{q^{a_\lambda(c)} - t^{l_\lambda(c)+1}} \prod_{c \in \mathcal{C}_{\lambda,\mu}} \frac{t^{l_\mu(c)} - q^{a_\mu(c)+1}}{t^{l_\lambda(c)} - q^{a_\lambda(c)+1}}, \quad c_{\lambda,\mu}(q, t) = \prod_{c \in \mathcal{R}_{\mu,\lambda}} \frac{t^{l_\mu(c)} - q^{a_\mu(c)+1}}{t^{l_\lambda(c)} - q^{a_\lambda(c)+1}} \prod_{c \in \mathcal{C}_{\mu,\lambda}} \frac{q^{a_\mu(c)} - t^{l_\mu(c)+1}}{q^{a_\lambda(c)} - t^{l_\lambda(c)+1}}$$

$\mathcal{R}_{\lambda,\mu}$  is the set of cells in the same row as  $\lambda/\mu$

$\mathcal{C}_{\lambda,\mu}$  is the set of cells in the same column as  $\lambda/\mu$

It is proven that  $\underline{e}_1 \left[ \frac{\mathbf{x}}{(1-q)(1-t)} \right]$  is the adjoint of  $\delta_{e_1}$  in respect to  $\langle \cdot, \cdot \rangle_*$ .

The Schur symmetric functions :

$$(75) \quad s_\mu := \sum_{\tau: \mu \rightarrow \mathbf{x}} x_\tau, \tau \text{ semi-standard and } x_\tau = \prod_{c \in \mu} x_{\tau(c)}$$

[Example 7 clic here](#)

Pieri formula :

$$(76) \quad h_n s_\mu = \sum_{\theta \vdash n+|\mu|} s_\theta, \quad (77) \quad e_n s_\mu = \sum_{\theta \vdash n+|\mu|} s_\theta.$$

$\theta/\mu$  ia a  $n$ -horizontal strip  $\theta/\mu$  ia a  $n$ -vertical strip

[Example 8 clic here](#)

The Kostka numbers  $K_{\mu,\lambda}$

$$(78) \quad K_{\mu,\lambda} := \#\{ \text{Semi-standard tableaux of shape } \mu \text{ fillings of } \lambda \}$$

$$(79) \quad s_\mu = \sum_{\lambda \vdash n} K_{\mu,\lambda} m_\lambda, \quad (80) \quad h_\mu = \sum_{\lambda \vdash n} K_{\lambda,\mu} s_\lambda, \quad (81) \quad e_\mu = \sum_{\lambda \vdash n} K_{\lambda,\mu} s_{\lambda'}$$

[Example 9 clic here](#)

domino tabloid,  $d_{\lambda,\mu}$

$$(82) \quad d_{\lambda,\mu} = \#\{ \text{domino tableaux of shape } \lambda \text{ and type } \mu \}$$

$$(83) \quad e_\lambda = \sum_{\mu \vdash n} (-1)^{|\mu| - \ell(\mu)} d_{\lambda,\mu} h_\mu, \quad (84) \quad h_\lambda = \sum_{\mu \vdash n} (-1)^{|\mu| - \ell(\mu)} d_{\lambda,\mu} e_\mu,$$

$\chi^\mu(\lambda)$ 

$$(85) \quad \chi^\mu(\lambda) := \sum_T (-1)^{ht(T)}, \text{ summed over all border-strip tableaux, } T, \text{ of shape } \mu \text{ and type } \lambda,$$

$$(86) \quad ht(T) = \prod ht(T_{\lambda_i}) \quad (87) \quad s_\mu = \sum_{\lambda \vdash n} \frac{1}{z_\mu} \chi^\mu(\lambda) p_\lambda$$

 $w_{\lambda,\mu}$  and  $v_{\lambda,\mu}$ 

$$(88) \quad w_{\lambda,\mu} = \#\{ \text{Matrices of zeros and ones, with row sums } \lambda \text{ and column sums } \mu \}$$

$$(89) \quad v_{\lambda,\mu} = \#\{ \text{Matrices of non negative integers, with row sums } \lambda \text{ and column sums } \mu \}$$

$$(90) \quad e_\lambda = \sum_{\mu \vdash n} w_{\lambda,\mu} m_\mu, \quad (91) \quad h_\lambda = \sum_{\mu \vdash n} v_{\lambda,\mu} m_\mu,$$

[Example 10 clic here](#)9.  $q$ -ANALOGS (SEE [BER2009] FOR MORE ON THIS)

$$(92) \quad [n]_q := 1 + q + q^2 + q^3 + \dots + q^{n-1} \quad (93) \quad [n]!_q := [n]_q [n-1]_q \dots [2]_q [1]_q$$

$$(94) \quad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q [n-k]!_q}$$

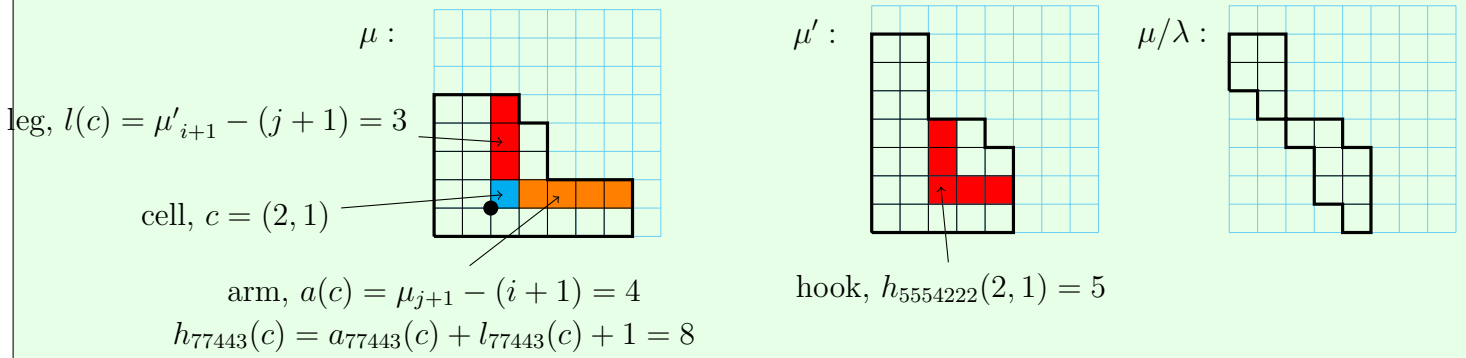
$$(95) \quad C_n(q) := \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q = \frac{[2n]_q [2n-1]_q \dots [n+3]_q [n+2]_q}{[n]_q [n-1]_q \dots [2]_q [1]_q}$$

$$(96) \quad e_k(1, q, q^2, \dots, q^{n-1}) = q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q \quad (97) \quad h_k(1, q, q^2, \dots, q^{n-1}) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q$$

[Example 11 clic here](#)



Example 1 :  $\mu = 77443 \vdash 25$ ,  $|77443| = 25$ ,  $\ell(77443) = 5$ ,  $\mu' = 5554222$  and  $\lambda = 43321$  :



$z_\mu = z_{77443} = 7^2 \cdot 4^2 \cdot 3 \cdot 2!2!1! = 9408$ ;  $n(\mu) = n(77443) = 39$ ;  $n(\mu') = n(5554222) = 57$ .  
[go back](#)

Example 2 :

Tableau

5	
3	63
111	9

shape 221  
No order

Semi-Standard Tableau

3	
2	3
1	1

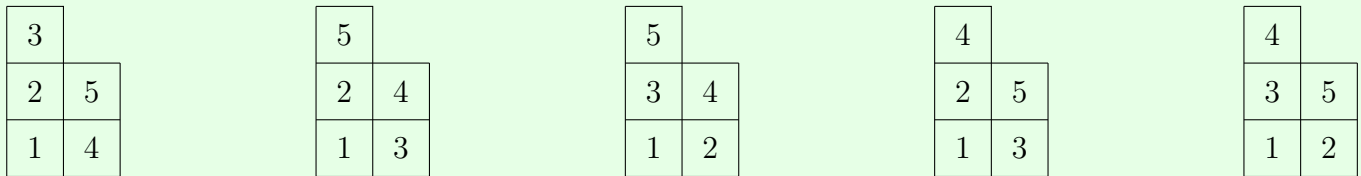
shape 221 and filling 212  
(i.e. filled by  $\{1^2, 2, 3^2\}$ )

Semi-Standard Tableau

3	
2	2
1	1

shape 221 and filling 221  
(i.e. filled by  $\{1, 1, 2, 2, 3\}$ )

All standard tableaux of shape 221 :



$$f^{221} = \frac{5!}{\prod_{c \in 221} h(c)} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 2 \cdot 3 \cdot 1 \cdot 1} = 5; \quad 5! = \sum_{\mu \vdash 5} (f^\mu)^2 = 1^2 + 4^2 + 5^2 + 6^2 + 5^2 + 4^2 + 1^2 = 120$$

[go back](#)

*Example 3 :*

$$m_{21}(x, y, z) = x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2$$

$$h_{21}(x, y, z) = h_2h_1 = (m_2 + m_{11})m_1 = (x^2 + y^2 + z^2 + xy + xz + yz)(x + y + z)$$

$$e_{21}(x, y, z) = (xy + xz + yz)(x + y + z)$$

$$p_{21}(x, y, z) = (x^2 + y^2 + z^2)(x + y + z)$$

$$s_{21}(x, y, z) = x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2 + 2xyz$$

For  $n = 4 :$

$$h_4(x, y, z) = m_{1111} + m_{211} + m_{22} + m_{31} + m_4$$

$$h_{31}(x, y, z) = 4m_{1111} + 3m_{211} + 2m_{22} + 2m_{31} + m_4$$

$$h_{22}(x, y, z) = 6m_{1111} + 4m_{211} + 3m_{22} + 2m_{31} + m_4$$

$$h_{211}(x, y, z) = 12m_{1111} + 7m_{211} + 4m_{22} + 3m_{31} + m_4$$

$$h_{1111}(x, y, z) = 24m_{1111} + 12m_{211} + 6m_{22} + 4m_{31} + m_4$$

$$s_4(x, y, z) = h_4$$

$$s_{31}(x, y, z) = h_{31} - h_4$$

$$s_{22}(x, y, z) = h_{22} - h_{31}$$

$$s_{211}(x, y, z) = h_{211} - h_{22} - h_{31} + h_4$$

$$s_{1111}(x, y, z) = h_{1111} - 3h_{211} + h_{22} + 2h_{31} - h_4$$

$$s_4(x, y, z) = e_{1111} - 3e_{211} + e_{22} + 2e_{31} - e_4$$

$$s_{31}(x, y, z) = e_{211} - e_{22} - e_{31} + e_4$$

$$s_{22}(x, y, z) = e_{22} - e_{31}$$

$$s_{211}(x, y, z) = e_{31} - e_4$$

$$s_{1111}(x, y, z) = e_4$$

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*Example 4 :*

a) Let  $f = (q + t)p_k$ , then :  $f \left[ \frac{5q\mathbf{x}}{1-t} \right] = (q + t)p_k \left[ \frac{5q\mathbf{x}}{1-t} \right] = (q + t) \frac{5q^n}{1-t^n} p_k(\mathbf{x})$ .

b)  $p_n[p_1(\mathbf{x})] = p_n[x_1 + x_2 + \dots] = p_n(\mathbf{x}) = \sum_{i \in \mathbb{N}} p_n[x_i] = \sum_{i \in \mathbb{N}} x_i^n$

c)  $p_n[p_k(\mathbf{x})] = p_n \left[ \sum_{i \in \mathbb{N}} x_i^k \right] = \sum_{i \in \mathbb{N}} p_n[x_i^k] = \sum_{i \in \mathbb{N}} x_i^{kn} = p_{nk}(\mathbf{x}) \Rightarrow p_n[f(\mathbf{x})] = f[p_n(\mathbf{x})] \forall f \in \Lambda$

d) Let  $g = p_3(\mathbf{x}) + p_{111}(\mathbf{x})$  and  $f = p_{11}(\mathbf{x}) + p_2(\mathbf{x})$ , then :

$$\begin{aligned} g[f(\mathbf{x})] &= (p_3 + p_{111})[f(\mathbf{x})] \\ &= p_3[p_{11}(\mathbf{x}) + p_2(\mathbf{x})] + p_{111}[p_{11}(\mathbf{x}) + p_2(\mathbf{x})] \\ &= p_3[p_{11}(\mathbf{x})] + p_3[p_2(\mathbf{x})] + (p_1[p_{11}(\mathbf{x}) + p_2(\mathbf{x})])^3 \\ &= p_3[p_1(\mathbf{x})]p_3[p_1(\mathbf{x})] + p_6(\mathbf{x}) + (p_{11}(\mathbf{x}) + p_2(\mathbf{x}))^3 \\ &= p_6(\mathbf{x}) + p_{33}(\mathbf{x}) + p_{222}(\mathbf{x}) + 3p_{2211}(\mathbf{x}) + 3p_{21111}(\mathbf{x}) + p_{1^6}(\mathbf{x}) \end{aligned}$$

and

$$\begin{aligned} f[g(\mathbf{x})] &= (p_{11} + p_2)[g(\mathbf{x})] \\ &= (p_1[p_3(\mathbf{x}) + p_{111}(\mathbf{x})])^2 + p_2[p_3(\mathbf{x}) + p_{111}(\mathbf{x})] \\ &= p_6(\mathbf{x}) + p_{33}(\mathbf{x}) + 2p_{3111}(\mathbf{x}) + p_{222}(\mathbf{x}) + p_{1^6}(\mathbf{x}) \end{aligned}$$

Therefore  $g[f(\mathbf{x})] \neq f[g(\mathbf{x})]$ .

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Example 5 :

$$\begin{aligned}
 H_4(x, y, z) &= q^6 s_{11111} + (q^5 + q^4 + q^3) s_{211} + (q^4 + q^2) s_{22} + (q^3 + q^2 + q) s_{31} + s_4 \\
 H_{31}(x, y, z) &= q^3 s_{11111} + (q^3 t + q^2 + q) s_{211} + (q^2 t + q) s_{22} + (q^2 t + qt + 1) s_{31} + t s_4 \\
 H_{22}(x, y, z) &= q^2 s_{11111} + (q^2 t + qt + q) s_{211} + (q^2 t^2 + 1) s_{22} + (qt^2 + qt + t) s_{31} + t^2 s_4 \\
 H_{211}(x, y, z) &= q s_{11111} + (qt^2 + qt + 1) s_{211} + (qt^2 + t) s_{22} + (qt^3 + t^2 + t) s_{31} + t^3 s_4 \\
 H_{1111}(x, y, z) &= s_{11111} + (t^3 + t^2 + t) s_{211} + (t^4 + t^2) s_{22} + (t^5 + t^4 + t^3) s_{31} + t^6 s_4
 \end{aligned}$$

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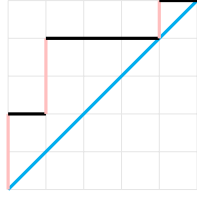


FIGURE 1.

Dyck path,  $\gamma \in \mathcal{D}_5$   
with riser  $\rho(\gamma) = 221$

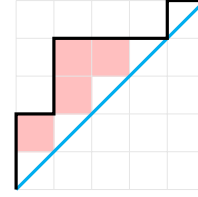
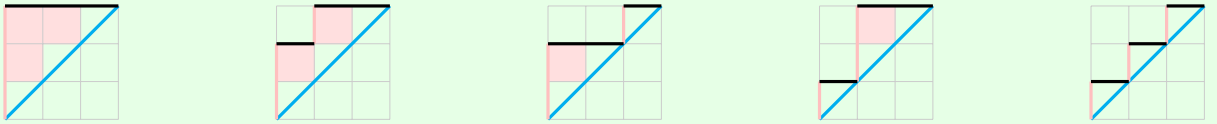


FIGURE 2.

Dyck path of area 4  
 $C_n := \#D_n$

Example 6 :



$$\nabla|_{t=1}(e_3) = q^3 e_3 + q^2 e_{21} + q e_{21} + q e_{21} + e_{111}$$

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Example 7 :

$$\begin{aligned}
 s_{21}(x, y, z) &= \begin{array}{|c|c|} \hline y & \\ \hline x & x \\ \hline \end{array} + \begin{array}{|c|c|} \hline z & \\ \hline x & x \\ \hline \end{array} + \begin{array}{|c|c|} \hline y & \\ \hline x & y \\ \hline \end{array} + \begin{array}{|c|c|} \hline z & \\ \hline x & z \\ \hline \end{array} + \begin{array}{|c|c|} \hline z & \\ \hline y & y \\ \hline \end{array} + \begin{array}{|c|c|} \hline z & \\ \hline y & z \\ \hline \end{array} + \begin{array}{|c|c|} \hline z & \\ \hline x & y \\ \hline \end{array} + \begin{array}{|c|c|} \hline y & \\ \hline x & z \\ \hline \end{array} \\
 &= x^2 y + x^2 z + x y^2 + x z^2 + y^2 z + y z^2 + 2 x y z
 \end{aligned}$$

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Example 8 :

$$\begin{aligned}
 h_3 s_{21} &= \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \times \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline \end{array} \\
 &= \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \\
 &= s_{51} + s_{411} + s_{42} + s_{321}
 \end{aligned}$$

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Example 9 :

$$\begin{aligned}
 K_{221,5} = 0, \quad K_{221,41} = 0, \quad K_{221,32} = 0, \quad K_{221,311} = 0, \quad K_{221,221} = 1, \quad K_{221,2111} = 2, \quad K_{221,11111} = 5 \\
 s_{221} = m_{221} + 2m_{2111} + 5m_{11111}
 \end{aligned}$$

$$K_{5,221} = 1, K_{41,221} = 2, K_{32,221} = 2, K_{311,221} = 1, K_{221,221} = 1, K_{2111,221} = 0, K_{11111,221} = 0$$

$$h_{221} = s_5 + 2s_{41} + 2s_{32} + s_{311} + s_{221}$$

$$e_{221} = s_{11111} + 2s_{2111} + 2s_{221} + s_{311} + s_{32}$$

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*Example 10 :*

$$w_{21,3} = \#\{\} = 0, w_{21,21} = \#\left\{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right\} = 1, w_{21,111} = \#\left\{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\} = 3$$

$$e_{21} = m_{21} + 3m_{111}$$

$$v_{21,3} = \#\left\{\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}\right\} = 1, v_{21,21} = \#\left\{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right\} = 2, v_{21,111} = w_{21,111} = 3$$

$$h_{21} = m_3 + 2m_{21} + 3m_{111},$$

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*Example 11 :*

$$[5]_q = q^4 + q^3 + q^2 + q + 1$$

$$[4]!_q = (q^3 + q^2 + q + 1)(q^2 + q + 1)(q + 1)(1)$$

$$\begin{bmatrix} 8 \\ 4 \end{bmatrix}_q = \frac{[8]_q [7]_q [6]_q [5]_q [4]_q [2]_q [2]_q [1]_q}{[4]_q [2]_q [2]_q [1]_q [4]_q [2]_q [2]_q [1]_q}$$

$$C_4(q) := \frac{1}{[5]_q} \begin{bmatrix} 8 \\ 4 \end{bmatrix}_q = \frac{[8]_q [7]_q [6]_q}{[4]_q [2]_q [2]_q [1]_q}$$

$$= 1 + q^2 + q^3 + 2q^4 + q^5 + 2q^6 + q^7 + 2q^8 + q^9 + q^{10} + q^{12}$$

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